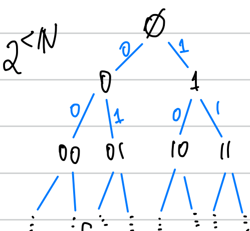
Math 592: Descriptive Set Theory
Lecture 2

2. Trees.

For a nonempty set A, we denote
$$A^{<\mathbb{N}} := \bigcup_{n \in \mathbb{N}} A^n,$$
where we think of elements of A^n as functions $n \to A$, where $n := \{0, 1, \dots, n-1\}$. Thus, $A^0 := \{\text{the unique function } \emptyset \to A\} = \{\emptyset\}$. Since functions are technically sets of pairs, for $s, t \in A^{<\mathbb{N}}$, it makes sense to write $s \subseteq t$, and this means that the function t extends s, i.e. $|s| \leq |t|$ and $s(i) = t(i)$ for all $i < |s|$, where $|s|$ denotes the size of the domain, called **length** of s.

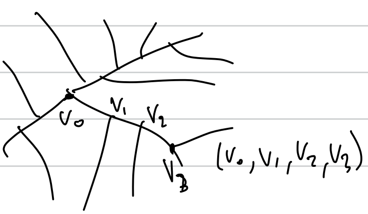
For $s \in A^{<\mathbb{N}}$ and $a \in A$, we write sa for the concatenation of s with a, i.e. $|sa| = |s|+1$ and $(sa)|_{|s|} = s$ and $sa(|s|) = a$. $\underbrace{\quad s \quad}\,\boxed{a}$.
Similarly, for $s, t \in A^{<\mathbb{N}}$, we write st for $\underbrace{\quad s \quad}\,\underbrace{\quad t \quad}$.

Picture. For $A := 2 = \{0, 1\}$.



Def. A subset $T \subseteq A^{<\mathbb{N}}$ is called a (set-theoretic) **tree** if $T \neq \emptyset$ and closed downward wrt $\subseteq$, i.e. if $s \subseteq t$ and $t \in T$, then $s \in T$. In particular, $\emptyset \in T$ because $T \neq \emptyset$.

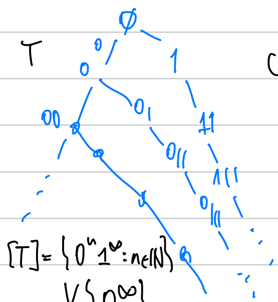
From a set-theoretic tree T, we may obtain a graph-theoretic tree G_T by taking T as the set of vertices and putting an edge $s - t$ exactly when $t = sa$ or $s = ta$ for some $a \in A$. Conversely, given a tree $G := (V, E)$ with a root v_0, we obtain a set theoretic tree T_G on the base set V by putting $s \in T \subseteq V^{<\mathbb{N}} \iff s$ is a path from v_0 to some $v \in V$. Note: $T_{G_T} \cong T$.



Infinite branches and closed subsets of $A^{\mathbb{N}}$.

For a tree $T \subseteq A^{<\mathbb{N}}$, let $[T]$ denote the set of infinite branches through T, i.e.

$$[T] := \{ x \in A^{\mathbb{N}} : x|_n \in T \ \forall n \in \mathbb{N} \} \subseteq A^{\mathbb{N}}.$$



$[T] = \{0^n 1^\infty : n \in \mathbb{N}\} \cup \{0^\infty\}$.

Conversely, given $Y \subseteq A^{\mathbb{N}}$, we get a tree:

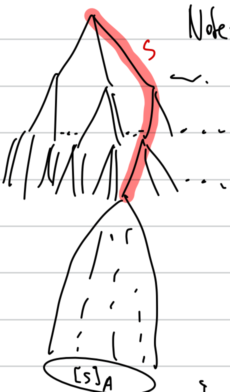
$$T_Y := \{ x|_n : x \in Y, n \in \mathbb{N} \}.$$

Note that T_Y is a pruned tree, i.e. for every $s \in T$ $\exists t \in T$ with $s \subsetneq t$.

Example. If $Y := \{0^n 1^\infty : n \in \mathbb{N}\}$, then $[T_Y] = Y \cup \{0^\infty\}$.

We give A the discrete top. In particular, $A^{\mathbb{N}}$ is Polish $\Leftrightarrow$ A is ctbl.
Then the cylinder sets: for $s \in A^{<\mathbb{N}}$,

$$[s]_A := \{ x \in A^{\mathbb{N}} : \underset{\text{x starts with the word s}}{x \supseteq s} \}.$$



Note: $[s]_A$ is clopen (i.e. closed and open) bc it's open by the def of product top and it is closed bc $[s]_A^c$ is open:

$$[s]_A^c = \bigcup_{\substack{t \in A^{|s|} \\ t \neq s}} [t]_A.$$

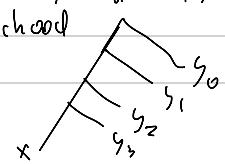
Here is a complete compatible metric d for $A^{\mathbb{N}}$:

$$d(x,y) := \begin{cases} 2^{-n} & \text{where } n \in \mathbb{N} \text{ is the least s.t. } x(n) \neq y(n) \\ 0 & \text{if } x = y. \end{cases}$$

In this metric, $B_{2^{-n}}(x) = [x|_n]$. For each $s \in A^{<\mathbb{N}}$, $[s]_A$ is the ball of radius $2^{-|s|}$ centered at any point of $[s]_A$ (the cantor's metric).

Prop. For any tree $T \subseteq A^{<\mathbb{N}}$, $[T]$ is closed. And for $Y \subseteq A^{\mathbb{N}}$, $[T_Y] = \overline{Y}$.
Thus, a set $Y \subseteq A^{\mathbb{N}}$ is closed iff $Y = [T_Y]$.

Proof. It is clear that $Y \subseteq [T_Y]$ by def. If $x \in [T_Y]$ then $\forall n \in \mathbb{N}$, $x|_n \in T_Y$, i.e. $\exists y_n \in Y$ s.t. $x|_n = y_n|_n$. But the sets $[x|_n]$ form a neighbourhood basis at x so $y_n \to x$ and thus $x \in \overline{Y}$. $\square$



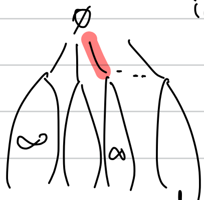
For a ctbl A, the spaces $A^{\mathbb{N}}$ are perhaps the most important Polish spaces, like $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$, because not only do they have a nice Polish zero-dim topology (i.e. admit a basis of clopen sets), but they also have combinatorics.

<u>Trees and compactness.</u> We already saw that closed subsets of $A^{\mathbb{N}}$ are exactly the sets of infinite branches through trees. But which of them are compact?

A tree $T \subseteq A^{<\mathbb{N}}$ is called finitely branching if every $s \in T$ has only finitely many immediate extensions, i.e. $\{a : sa \in T, a \in A\}$ is finite, i.e. every vertex has finite degree.

<u>Kőnig's lemma.</u> Every finitely branching infinite tree $T \subseteq A^{<\mathbb{N}}$ has an infinite branch, i.e. $[T] \neq \emptyset$.

Proof.



Prop. For any pruned tree $T \subseteq A^{<\mathbb{N}}$, $[T]$ is compact $\iff$ T is finitely branching. In particular, $Y \subseteq A^{\mathbb{N}}$ is compact $\iff$ T_Y is finitely branching.